Stability of multiple pulses in discrete systems

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The stability of multiple-pulse solutions to the discrete nonlinear Schrödinger equation is considered. A bound state of widely separated single pulses is rigorously shown to be unstable, unless the phase shift $\Delta\phi$ between adjacent pulses satisfies $\Delta\phi=\pi$. This instability is accounted for by positive real eigenvalues in the linearized system. The analysis leading to the instability result does not, however, determine the linear stability of those multiple pulses for which $\Delta\phi=\pi$ between adjacent pulses. A direct variational approach for a two-pulse predicts that it is linearly stable if $\Delta\phi=\pi$, and if the separation between the individual pulses satisfies a certain condition. The variational approach can easily be generalized to study the stability of N pulses for any $N\geqslant 3$. The analysis is supplemented with a detailed numerical stabilty analysis.

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I. INTRODUCTION

The importance of differential-difference equations (DDEs) for modeling various physical phenomena, as well as their general interest as dynamical systems, has become evident in recent years. A ubiquitous DDE is the generalized discrete nonlinear Schrödinger equation (DNLS) equation; see, e.g., Refs. [1,2]. Most relevant for realistic applications (particularly to nonlinear optics) is the DNLS equation with the cubic (Kerr) nonlinearity [3–8]:

$$i\dot{u}_n+(1/2)\Delta_2 u_n+|u_n|^2u_n=0, \tag{1.1}$$

where the overdot stands for the differentiation in time t, u_n are complex variables defined for all integer values of the site index n, and Δ_2 is the second order difference operator with spacing h:

$$\Delta_2 u_n=h^{-2}(u_{n+1}+u_{n-1}-2u_n),\quad h>0.$$

The DNLS equation was used by Aceves *et al.* [3,4] to model the propagation of discrete self-trapped beams in an array of linearly coupled nonlinear optical waveguides. Experimental results for optical waveguide arrays confirming the validity of the model have recently been reported by Morandotti *et al.* [8] and by Eisenberg *et al.* [5]. It is also often used as an envelope equation modeling the local denaturation of the DNA double strand [9].

Much theoretical work on the DNLS equation has been done concerning the existence of steady state solutions (see [10–14,1] and references therein). In particular, Ref. [1] provides existence and stability conditions for solitons in the DNLS equation with an arbitrary power on-site nonlinearity, rather than limiting the analysis to the purely cubic case. The interested reader should also consult Hennig [15] and Kollmann *et al.* [16] for existence results concerning the driven and damped DNLS. Very recently, by exploiting the fact that the DNLS is a Hamiltonian system that conserves the l^2 norm of solutions, Weinstein [17] has shown that the system possesses a minimizer that is a time-independent solution. Furthermore, he was able to characterize that minimizer in the anticontinuum limit ($h\to+\infty$). These minimizers represent the so-called one-site breathers (see Johansson and Aubry [12] and MacKay and Aubry [18]). However, no such characterization was given for $0<h\lesssim O(1)$; in particular, the structure of the minimizer in the continuum limit ($h\to 0$) was unknown. The existence and stability of solutions for small h was considered by Kapitula and Kevrekidis [19]. For small h it was seen that the DNLS has two pulse solutions, viz., a stable site-centered one which is approximately given by $U_n=\sqrt{2}\,\mathrm{sech}(\sqrt{2}hn)e^{it}$, and an unstable intersite-centered solution which is roughly $U_n=\sqrt{2}\,\mathrm{sech}[\sqrt{2}h(n+1/2)]e^{it}$. The instability is accounted for by a positive real eigenvalue of $O(\exp(-\pi^2/\sqrt{2}\,h))$ for the linearized system. It should be remarked that this stability result is also given by Laedke *et al.* [1] in the case that the perturbation of the wave is parity preserving.

Let $U_n^{(1)}$ represent a single-humped soliton solution to Eq. (2.4). From the work of Bountis *et al.* [10] it is known that, if this pulse is constructed as a transversal intersection of stable and unstable manifolds for the steady-state dynamical system, then there exist k-soliton solutions $U_n^{(k)}$. Each of these solutions is realized as a bound state of k widely separated copies of $U_n^{(1)}$. Furthermore, each bound-state solution $U_n^{(k)}$ has k different versions, with the number of phase jumps between adjacent humps, i.e., the number of sign changes inside the solution, taking all the integer values from 0 to $k-1$. Kapitula and Kevrekidis [19] showed that this transversality condition is satisfied in Eq. (2.4) for small h.

The goal in this paper is to determine the stability of these k solitons for each $k\geqslant 2$. In Sec. II, a rigorous instability criterion will be given, showing that a majority of the k solitons represent unstable configurations, with the instability being induced by real positive eigenvalues in the linearized problem. In Sec. III, considerations within the quasicontinuum approximation for $k=2$ will show that the bound states with alternating signs of the single-humped pulses can indeed be dynamically stable. Lastly, in Sec. IV we display the results of direct numerical computations that corroborate the analytical predictions.

Before continuing, let us note that bound states of solitons, in the form of so-called breathers, are also known in the continuum NLS equation, where they are available as exact solutions produced by the inverse scattering transform, or may be obtained perturbatively [20,21]. A principal difference from the discrete model, in which static bound states of solitons are possible, is that in the continuum model the coupled solitons oscillate, periodically passing through each other. Another difference is that, as we will demonstrate in the present work, the static bound state of discrete solitons may be completely stable, essentially due to to the existence of a finite binding energy, while the bound states of solitons in the exactly integrable continuum NLS equation have their binding energy exactly equal to zero (see Ref. [22]); hence they all are unstable. The method of Refs. [20,21] has also been generalized to the problem of *N*-solitons and their interaction and analogs with the complex Toda chain have been drawn in Refs. [23,24]. Results for multisoliton complexes also exist for the damped and driven continuum NLS (see, e.g., Ref. [25]). Static bound states of solitons may exist in continuum dissipative models of the Ginzburg-Landau type, but consideration of dissipative models is beyond the framework of the present work.

II. INSTABILITY RESULT

The DNLS equation can be derived from the Hamiltonian

$$H=\frac{1}{2}\sum_{n=-\infty}^{+\infty}(h^{-2}|u_n-u_{n-1}|^2-|u_n|^4) \tag{2.1}$$

via $\dot{u}_n=i\partial H/\partial\overline{u_n}$ (the overbar denotes the complex conjugate). In addition to the Hamiltonian, another dynamical invariant preserved by the equation is the power, or "number of particles,"

$$P=\frac{1}{2}\sum_{n=-\infty}^{+\infty}|u_n|^2. \tag{2.2}$$

An equation for stationary solutions U_n can be derived as $\partial H/\partial\overline{u}_n-\partial P/\partial\overline{u}_n=0$, i.e.,

$$(1/2)\Delta_2 U_n-U_n+|U_n|^2U_n=0. \tag{2.3}$$

The multibump solutions $U_n^{(k)}$ are then time-independent solutions of

$$i\dot{u}_n+(1/2)\Delta_2 u_n-u_n+|u_n|^2u_n=0. \tag{2.4}$$

Linearizing Eq. (2.4) about the stationary solution $U_n^{(k)}$ produces a linear operator

$$L^{(k)}=J\begin{bmatrix}L_+^{(k)} & 0\\ 0 & L_-^{(k)}\end{bmatrix},\quad J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tag{2.5}$$

where the auxiliary operators are defined as

$$L_+^{(k)}q_n=\frac{1}{2}\Delta_2 q_n-q_n+6(U_n^{(k)})^2q_n,$$

$$L_-^{(k)}r_n=\frac{1}{2}\Delta_2 r_n-r_n+2(U_n^{(k)})^2r_n. \tag{2.6}$$

It is clear that the operators $L_\pm^{(k)}$ are self-adjoint; hence they each have only a real spectrum. Furthermore, since the DNLS equation is a Hamiltonian system, the eigenvalues of the full operator $L^{(k)}$ satisfy the restriction that if λ is an eigenvalue then so also are $-\lambda$ and $\pm\overline{\lambda}$.

Set

$$p(L_\pm^{(k)})=\{\lambda\in\sigma(L_\pm^{(k)}):\lambda>0\}, \tag{2.7}$$

i.e., $p(L_\pm^{(k)})$ is the number of positive eigenvalues of the operator $L_\pm^{(k)}$. Following the work of Grillakis *et al.* [26–28], it is known that if $|p(L_+^{(k)})-p(L_-^{(k)})|>1$ then the solution $U_n^{(k)}$ is unstable, and the instability is manifested in the presence of one or more real positive eigenvalues of the operator $L^{(k)}$. Let

$$N_s=\text{number of times } U_n^{(k)} \text{ changes sign}, \tag{2.8}$$

and note that $0\leq N_s\leq k-1$. It is easy to check that $L_-^{(k)}(U_n^{(k)})=0$. As a consequence, by applying Sturm-Liouville theory one gets that

$$p(L_-^{(k)})=N_s$$

(see Levy and Lessman [29] on the applicability of Sturm-Liouville theory to difference equations). It is relatively straightforward to check that $L_+^{(1)}$ has at least one positive eigenvalue (for example, see [1]). It was shown by Kapitula and Kevrekidis [19] that for h sufficiently small $L_+^{(1)}$ may have at most one other positive eigenvalue. As a consequence of the work of Alexander and Jones [30,31] and Sandstede [32], one concludes that, if $U_n^{(k)}$ is constructed as k widely separated copies of $U_n^{(1)}$, then there are k eigenvalues of $L_+^{(k)}$ near each positive eigenvalue of $L_+^{(1)}$, i.e.,

$$p(L_+^{(k)})\geq k.$$

It is thus seen that if $0\leq N_s<k-1$ then the linear operator $L^{(k)}$ has at least one positive real eigenvalue, and hence the solution is unstable.

It should be remarked that the condition $|p(L_+^{(k)})-p(L_-^{(k)})|=1$ does not necessarily imply that the wave is linearly stable. The eigenvalue problem can be rewritten as

$$[L_-^{(k)}\lambda^2(L_+^{(k)})^{-1}]p=0. \tag{2.9}$$

If λ_0 is an eigenvalue for Eq. (2.9) with corresponding eigenfunction p_0, then the Krein sign of λ_0 is given by $s(\lambda_0)$

$=-\mathrm{sgn}\langle (L_+^{(k)})^{-1}p_0, p_0\rangle$, where $\langle f,g\rangle = \Sigma_{n=-\infty}^{+\infty} f_n g_n$. Under the assumption that $|p(L_+^{(k)})-p(L_-^{(k)})|=1$, it was shown by Grillakis [27] that there are non-negative integers n_r and n_i with $n_r+n_i=k-1$ such that the operator $L^{(k)}$ has n_r eigenvalues of negative Krein sign and n_i complex eigenvalues with nonzero real part. The eigenvalues of negative Krein sign are either purely real and positive or purely imaginary but *structurally unstable*. If the eigenvalue is structurally unstable, then a small perturbation of the vector field can eject it off the imaginary axis, leading to an unstable eigenvalue with positive real part. Furthermore, if eigenvalues of opposite sign collide, then they will generically form a complex conjugate pair after the collision, whereas if eigenvalues of the same sign collide, then they will just pass through each other. The interested reader should consult Grillakis [27] and Li and Promislow [33,34] for further details and examples.

In the case of a two-pulse bound state, which will be discussed in the next section, if $|p(L_+^{(2)})-p(L_-^{(2)})|=1$, then $n_r+n_i=1$. Hence, if a linear instability arises it must do so through either the appearance of one real pair of eigenvalues, $\{\pm\lambda_r\}$, or via a quadruplet of complex eigenvalues with a nonzero real part $\{\pm\lambda_c, \pm\overline{\lambda_c}\}$. One must do further analysis to determine if the solution is truly linearly stable.

III. STABILITY OF TWO SOLITONS

The objective now is to consider the interaction between solitons and to determine if there can be stable bound states in the event $N_s=k-1$, i.e., when adjacent fundamental pulses have alternating signs. This can be done analytically in the quasicontinuum approximation, when the pulse is approximated by the exact one-soliton solution to the continuum NLS equation. The latter equation is

$$i\partial_t u + (1/2)\partial_x^2 u + |u|^2 u = 0, \tag{3.1}$$

and it is related to Eq. (1.1) by $x \equiv hn$, $h\to 0$. Thus, we adopt the following approximation for the pulse:

$$u_n = \eta\,\mathrm{sech}[\eta(hn-\xi)]\exp[i\phi(t)], \tag{3.2}$$

where η is the amplitude, η^{-1} is the size, ξ is the coordinate for the center, and $\phi(t)=(1/2)\eta^2 t+\phi_0$, where ϕ_0 is an arbitrary phase constant. The quasicontinuum approximation assumes that $\eta h \ll 1$, i.e., the size of the soliton (pulse) is much larger than the lattice spacing.

We will perform the stability analysis for the case $k=2$, i.e., for two widely separated near-identical pulses whose centers are placed at points $\xi_{1,2}$, and whose phase difference is $\Delta\phi\in\{0,\pi\}$. All that follows below can easily be generalized for $k\geq 3$, although this will not be done here. In the quasicontinuum approximation, it is easy to derive an effective potential of interaction between the solitons (see., e.g., Ref. [35]):

$$U_{\mathrm{int}}(\xi_1-\xi_2,\Delta\phi) = -8\eta^3\exp(-\eta|\xi_1-\xi_2|)\cos(\Delta\phi). \tag{3.3}$$

The assumption that the solitons are far separated implies that $\eta|\xi_1-\xi_2|\gg 1$. It immediately follows from the expression (3.3) that a stationary bound state of the two solitons must have $\partial U_{\mathrm{int}}/\partial(\Delta\phi)=0$, i.e., $\Delta\phi=0$ or $\Delta\phi=\pi$. In Ref. [35] it has been shown that a *negative* effective mass corresponds to the phase degree of freedom of the two-soliton bound state in the continuum approximation. Hence, for such a negative mass only the maximum of U_{int} (i.e., the state with $\Delta\phi=\pi$) is potentially stable (in contrast to what would be true for a positive mass), in accordance with the general rigorous result obtained in the previous section. Therefore, in what follows below we will set $\Delta\phi=\pi$.

The next step is to find a part of the Hamiltonian associated with the discrete soliton interaction. Unlike the expression (3.3), it will actually take the discreteness of the system into account. The calculation is based on the expression for the Hamiltonian given by Eq. (2.1), and makes use of the formula

$$\sum_{n=-\infty}^{+\infty}\exp(i\alpha n) = 2\pi\sum_{m=-\infty}^{+\infty}\delta(\alpha-2\pi m), \tag{3.4}$$

where α is an arbitrary real parameter. Upon substituting the approximate soliton shape of Eq. (3.2) into Eq. (2.1), one obtains, at the lowest-order approximation in the small parameter ηh,

$$H_{\mathrm{sol}}(\xi) \approx -\frac{8\pi^4}{3h^3}\exp\left(-\frac{\pi^2}{\eta h}\right)\cos\left(\frac{2\pi}{h}\xi\right). \tag{3.5}$$

Note that the coefficient $\exp(-\pi^2/\eta h)$ in the expression (3.5) is *exponentially small*, due to the assumed smallness of ηh. Similar exponentially small terms due to discreteness have also been recently observed in works by Kevrekidis *et al.* [36] and by Kapitula *et al.* [19,37].

The net effective Hamiltonian for a two-soliton state is the sum of the expressions (3.3) and (3.5), the latter being taken separately for each soliton:

$$\begin{aligned} H^{(2)} &= H_{\mathrm{sol}}(\xi_1)+H_{\mathrm{sol}}(\xi_2)+U_{\mathrm{int}}(\xi_1-\xi_2) \\ &= -\frac{16\pi^4}{3h^3}\exp\left(-\frac{\pi^2}{\eta h}\right)\cos\left(\frac{2\pi}{h}Z\right)\cos\left(\frac{2\pi}{h}\Delta\xi\right) \\ &\quad + 8\eta^3\exp(-2\eta\Delta\xi), \end{aligned} \tag{3.6}$$

where we define $\Delta\xi\equiv(1/2)(\xi_1-\xi_2)$ and $Z\equiv(1/2)(\xi_1+\xi_2)$, which are the separation between the solitons in the bound state, and the coordinate of the bound state's ''center of mass'' relative to the underlying lattice. It is clear that the above procedure can be generalized to create an effective Hamiltonian $H^{(k)}$ for any $k\geq 3$.

Stationary boundary states of the two solitons correspond to fixed points (FPs) of the Hamiltonian (2.1), which are defined by the obvious equations $\partial H^{(2)}/\partial Z=0$ and $\partial H^{(2)}/\partial\Delta\xi=0$. Because the effective masses corresponding to the solitons' coordinates $\xi_{1,2}$ are positive [35], the stability of these FPs is determined by the standard conditions stating that the function $H^{(2)}$ of the two variables Z and $\Delta\xi$ must have a local *minimum* at a FP, i.e., the values of the partial second derivatives of $H^{(2)}$ at a FP must constitute a positive-definite quadratic form.

It is straightforward to see that there are two types of FPs with $\sin(2\pi Z/h)=0$. One is given by

$$Z=0,\quad \frac{2\pi^5}{3(\eta h)^4}\exp\left(-\frac{\pi^2}{\eta h}\right)\sin\left(\frac{2\pi}{h}\Delta\xi\right)=\exp(-2\eta\Delta\xi), \tag{3.7}$$

and the other one satisfies

$$Z=\frac{h}{2},\quad \frac{2\pi^5}{3(\eta h)^4}\exp\left(-\frac{\pi^2}{\eta h}\right)\sin\left(\frac{2\pi}{h}\Delta\xi\right)=-\exp(-2\eta\Delta\xi). \tag{3.8}$$

Obviously, the bound states corresponding to these two types of FP can be classified, relative to the underlying lattice, as having their center of mass either site centered on intersite centered, respectively. Additionally, there exists another set of FPs with $\cos(2\pi\Delta\xi/h)=0$, but one can immediately check that those can never realize a minimum or maximum of $H^{(2)}$. Instead, they are saddle points, and hence always unstable. Therefore, they are not considered in what follows below.

In order to better understand the effect of the separation distance on the stability of the wave, we set $\Delta\xi=mh+\nu$, where m is a positive integer and $0\leqslant\nu<h$. Continuing the consideration of the fixed points, we notice that in the lowest-order approximation in the small parameter ηh, Eqs. (3.7) and (3.8) yield the following minimum separation between the solitons in the bound state:

$$m\geqslant m_{\min}>(1/2)(\pi/\eta h)^2. \tag{3.9}$$

Beyond this minimum distance, there exists an infinite set of bound states with larger separations [35]. Using the fact that $m_{\min}$ is large allows one to see that there are two different solutions, in which the residual contribution ν to $\Delta\xi$ is close to either $\nu=0$ or $\nu=1/2$. This implies that for m sufficiently large one has $\Delta\xi=mh$ or $\Delta\xi=(m+1/2)h$. It can then be checked, upon considering the positive definiteness of the above-mentioned set of second derivatives of Eq. (3.6) at each FP, that for FP (3.7) the stable solution corresponds to $\nu=0$, whereas for FP (3.8) the stable one is with $\nu=h/2$.

Finally, it is natural to enquire as to what the stable two-pulse looks like. Assume that $m\gg m_{\min}$. It is shown in [19] that a stable one-pulse is site centered ($\mathcal{S}$), whereas an unstable one-pulse is intersite centered ($\mathcal{I}$). Under the assumption of the separation distance, it is known that each of the individual pulses associated with the two-pulse is a small perturbation of a one-pulse [38]. Hence, one can construct a two-pulse via the combinations $\mathcal{SS}$, $\mathcal{SI}$, $\mathcal{IS}$, or $\mathcal{II}$. From the work of Sandstede [32] it is known immediately that any construction that involves $\mathcal{I}$ will be unstable. As a consequence, for sufficiently large m the local minimum of $H^{(2)}$ corresponds to a two-pulse of the type $\mathcal{SS}$, while the saddle point corresponds to either $\mathcal{SI}$ or $\mathcal{IS}$, and the local maximum is the two-pulse $\mathcal{II}$.

IV. NUMERICAL RESULTS

It is very desirable to compare the above analytical results with numerical experiments. In this section, we report on results of simulations of two-pulse bound states whose separation distance satisfies $m>m_{\min}$. It is well known that the linearization of the continuum NLS equation (3.1) around the soliton (3.2) yields four zero eigenvalues, the so-called Goldstone modes. These eigenvalues are a consequence of the translational and rotational symmetries associated with the continuum NLS. Notice, however, that here we will use the term Goldstone in particular for eigenvalues that are related to the continuum problem's translational invariance symmetry. When linearizing the DNLS equation around the solution $U_n^{(1)}$ there will continue to be four eigenvalues near the origin [19]. Two of these eigenvalues are exactly equal to zero, due to the rotational invariance associated with the DNLS equation, while the other two become exponentially small, mirroring the exponentially small splitting of the homoclinic orbits [19,36]. It should be noted that hereafter, by convention, such modes that bifurcate away from zero will still be called Goldstone modes (even though the discrete system does not respect the continuum symmetry). Following Ref. [32], it is known that for the two-soliton there are eight eigenvalues near the origin: two equal to zero, four Goldstone ones, and two more that are close to the origin and of order $O(\exp(-Cmh))$ for some $C>0$. These final two eigenvalues arise from the tail-mediated interaction of the two far-separated solitons.

Since the results presented by Eqs. (3.7) and (3.8) can be expressed in terms of the rescaled variable ηh, from here on we will set $\eta=1$ without loss of generality. In order to compute solutions to Eq. (1.1), we used the ansatz $u_n(t)=\exp(it/2)\psi_n(t)$ and solved the ensuing nonlinear algebraic equations by means of Newton's method with suitable initial conditions. To perform a stability analysis of the obtained solutions, we then consider a perturbed solution

$$u_n(t)=\exp(it/2)[U_n^{(2)}+v_n(t)],$$

with $v_n=a_n\exp(-i\gamma t)+b_n\exp(i\gamma t)$. The solution is linearly unstable if there is an eigenvalue $\lambda=i\gamma$ with $\mathrm{Im}\,\gamma\neq 0$, and linearly stable if γ is real. The above ansatz leads to the linearized equations

$$\gamma a_n=-\Delta_2 a_n-2|U_n^{(2)}|^2 a_n-a_n/2-(U_n^{(2)})^2 b_n^{\star}, \tag{4.1}$$

$$-\gamma b_n=-\Delta_2 b_n-2|U_n^{(2)}|^2 b_n-b_n/2-(U_n^{(2)})^2 a_n^{\star}. \tag{4.2}$$

We then use Eq. (4.1) and the complex conjugate of Eq. (4.2) to solve the resulting matrix eigenvalue problem for $(\gamma,\{a_n,b_n^{\star}\})$. Finally, we complement our stability analysis with direct numerical simulations of the system, using a fourth-order explicit Runge-Kutta integrator.

In all that follows, we call the two-pulse solution with the phase difference $\Delta\phi=0$ an *up-up soliton*, and that with $\Delta\phi=\pi$ an *up-down soliton*. The following results have been obtained.

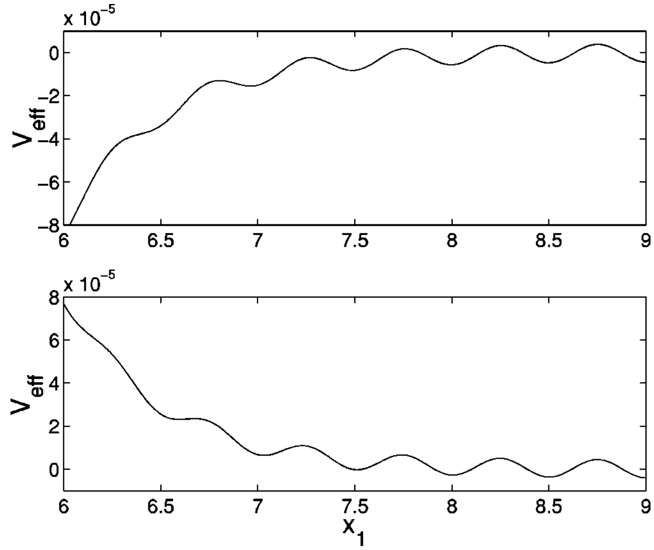

FIG. 1. Plot of the effective potential as a function of the position of the first pulse from the center $x_1=0$ for $h=0.5$ and for the up-up (top panel) and the up-down (bottom panel) pulse configurations. The position of the second pulse is $x_2=-x_1+h$.

(1) Neglecting, for the moment, the case of the saddle-point configurations, there are four principal possibilities for the two-pulse solutions. In particular, the numerically computed interaction potential between fundamental pulses forming the up-up soliton, and its counterpart for the up-down soliton, are displayed in Fig. 1. Since in each case there are local minima and maxima of the effective potential, four configurations are possible, whose stability we will consider below: up-up at a local minimum, up-up at a local maximum, up-down at a local minimum, and up-down at a local maximum.

(2) The up-up solitons are *always unstable*, in agreement with the predictions made in Secs. II and III. In this case, the tail-tail interaction is attractive, inducing an unstable eigenmode with γ purely imaginary. The behavior of the Goldstone eigenvalues is determined by local features of the effective potential. In particular, for a local maximum of the potential the γ_g's (the subscript g will be used for Goldstone eigenmodes) are imaginary (see Fig. 2), while in case of the local minimum case they are real (see Fig. 3). It should be stressed here that merely looking at the local picture of the effective potential could lead to false conclusions in cases like the one at hand. Even though the existence of a local minimum implies the stability of the Goldstone modes, the stability in the full infinite-dimensional system is actually determined by the unstable tail-tail interaction modes, and hence turns out to be opposite to that obtained by naively observing the effective potential.

(3) On the contrary, the up-down pulses *can* be linearly stable. In this case, the repulsive tail-tail interaction induces an interaction mode with a real γ. Thus, in this case stability may indeed be based upon observation of the local features of the effective potential. For the case of local maxima, the γ_g's are imaginary, signaling the instability of the configuration (Fig. 4); however, for the local minima all the γ's are real (Fig. 5); hence the configuration is linearly stable. We

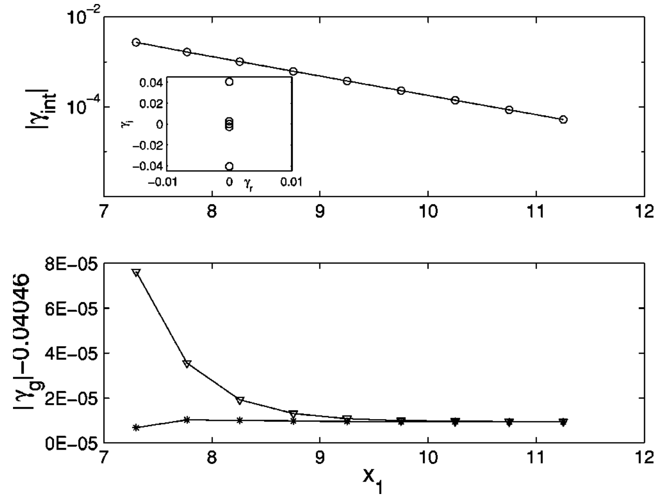

FIG. 2. Plot of the γ's for the up-up configuration at a local maximum of the interaction potential at $h=0.5$. Shown are two pairs of the Goldstone-mode γ's (stars and triangles in the bottom subplot), and the the one corresponding to the interaction mode (circles in the top subplot), as a function of the position of the first pulse x_1, while $x_2=-x_1+h$. Notice that for the eigenvalues $\lambda=i\gamma$; hence when γ is imaginary (in which case it will be denoted by the absolute value), the corresponding mode induces instability. In contrast, when all the γ's are real (in which case they will appear without absolute values), the configuration will be stable. The same notation is followed in Figs. 3–5, below. Note a clear exponential dependence of the interaction-mode frequency on x_1.

conclude that the only stable configuration among the four possible ones is of the up-down type with a local minimum, in full accord with the analytical results obtained in Sec. III. It should also be remarked here that in all four cases the absolute value of the γ of the tail-tail interaction eigenmode can be very clearly seen to decay exponentially as a function of the distance between the pulse centers, in agreement with the theoretical prediction mentioned above.

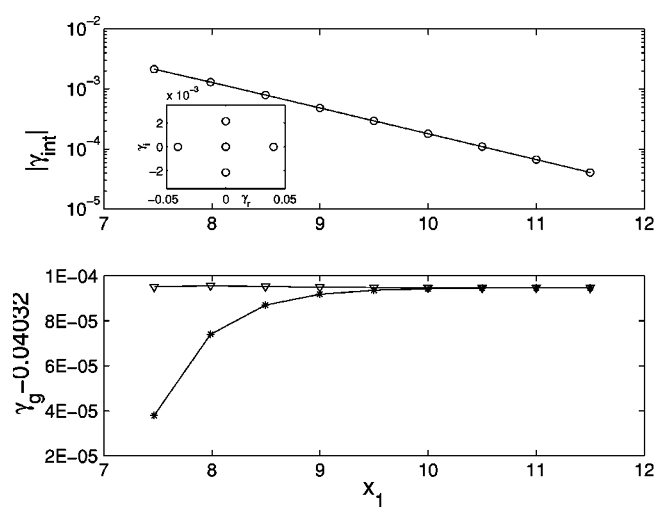

FIG. 3. γ's for the up-up configuration at a local minimum of the potential. As above, the interaction-mode eigenvalue signals instability, but the Goldstone γ's are real in this case (as necessitated by the local curvature of the potential barrier near the minimum). The notation is the same as in Fig. 2.

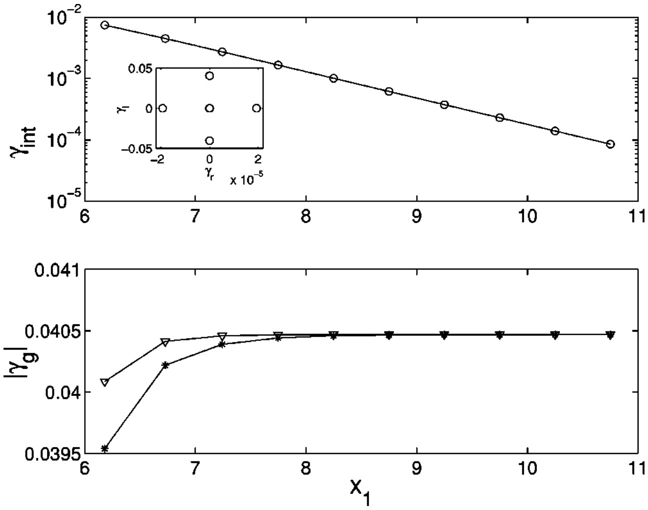

FIG. 4. γ's of the up-down configuration at a local maximum. The tail mode is now stable, but the instability comes through the unstable Goldstone pairs. The notation is the same as in Fig. 2.

In fact, a precursor suggesting the stability of the up-down configurations under suitable conditions in the DNLS equation was the study of *twisted localized modes* reported in [39,40]. The localized structures studied therein can be thought of as a special case of very closely placed up-down pulses. Hence, their stability, identified for sufficiently small h in these works, can be put in the same general context as presented herein.

(4) In order to observe how the above-mentioned instabilities manifest themselves in the dynamics of the full system, we initialize the system at an unstable configuration and add a small amount of random noise to the initial condition. For an initial up-up configuration, when the solitons are sufficiently far apart a typical evolution results in turning the original up-up configuration into an up-down one. The up-down pulse then oscillates around a stable minimum of its interaction potential. In contrast, for an initial up-down configuration the typical evolution involves oscillations of the original local maximum configuration around one of a nearby local minimum, which is chosen according to a ''push'' given by the random perturbation added to the initial conditions.

When the solitons are initially very close, in which case our theory is not strictly valid but still turns out to be qualitatively correct, atypical behavior may occur. In particular, for up-up configurations the attractive interaction may be so strong that it results in collision between the fundamental pulses, while for the up-down configurations the repulsion of the solitons, when starting at the first maximum of the potential in Fig. 1 (the one closest to $x=0$), may result in eventual separation of the solitons (corresponding to the soliton sliding down the potential curve in Fig. 1). This can be seen to occur in Fig. 6.

(5) We have so far examined possible configurations and their stability, as well as the variation of this behavior as a function of the distance between the fundamental pulses. Now we turn our attention to the change of the behavior with the lattice spacing. Obviously, stationary bound states of fundamental pulses exist only in the discrete system, disappearing in the continuum limit. However, as we approach the anticontinuum limit, $h\rightarrow\infty$, there is no reason for the solutions to disappear. Instead, they fit very naturally into the anticontinuum-limit picture set forth by Aubry *et al.* [12,18]. We trace this, varying h for a specific up-down solution. For large h, the behavior of the γ's, all of which are real, is quite smooth. Eventually, all the γ's of the localized eigenmodes, except for those corresponding to the tail-tail interaction, will merge with the continuous spectrum (phonon band, see Fig. 7). This includes the pairs bifurcating from the band edge, as well as the Goldstone pairs [41,19]. It should be noted that the possibility of a pair of eigenvalues bifurcating from the edge of the phonon band was shown by Alexander and Jones [30,31], by Gardner and Zumbrun [42], and by Kapitula and Kevrekidis [19]. Decreasing h, we observe that the configuration initially becomes unstable through the bifurcation of a pair along the imaginary axis (around $h\approx 0.45$), and eventually for smaller h the branch terminates in a saddle-node bifurcation (at $h\approx 0.355$). This saddle-node bifurcation is consistent with the picture presented by Bountis *et al.* [10].

(6) For the saddle configurations referred to in Sec. III, we also used our technique to identify them and study their linear stability. In that case, just as expected, given the nature ($\mathcal{IS}$) of the saddle point, one pair of γ_g's is always imaginary, while the other one is real. The position of the tail-interaction pair is once again dictated by the up-up or up-down nature of the configuration in a manner similar to the previously considered cases. Figure 8 shows the spectral plane close to the origin for such a saddle-point configuration.

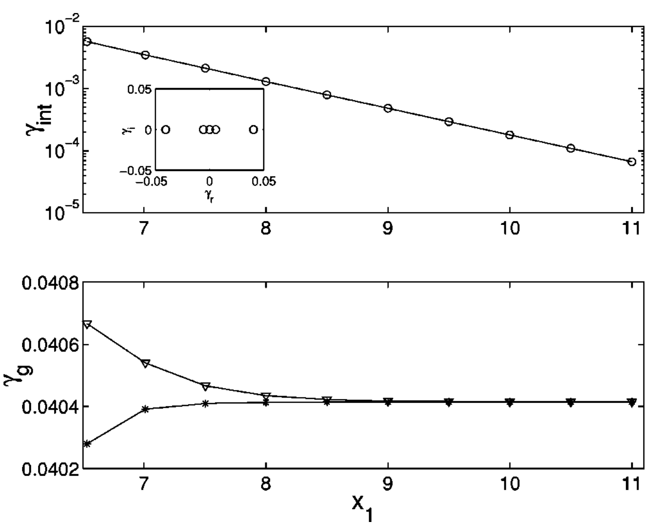

FIG. 5. γ's of the up-down state at a local minimum. All the γ's are real, and the configuration is linearly stable. The notation is the same as in Fig. 2.

V. CONCLUSIONS

As far as we know this is the first piece of analytical work that deals with the stability of multiple pulses to the DNLS; however, there has been a great deal of work concerning the

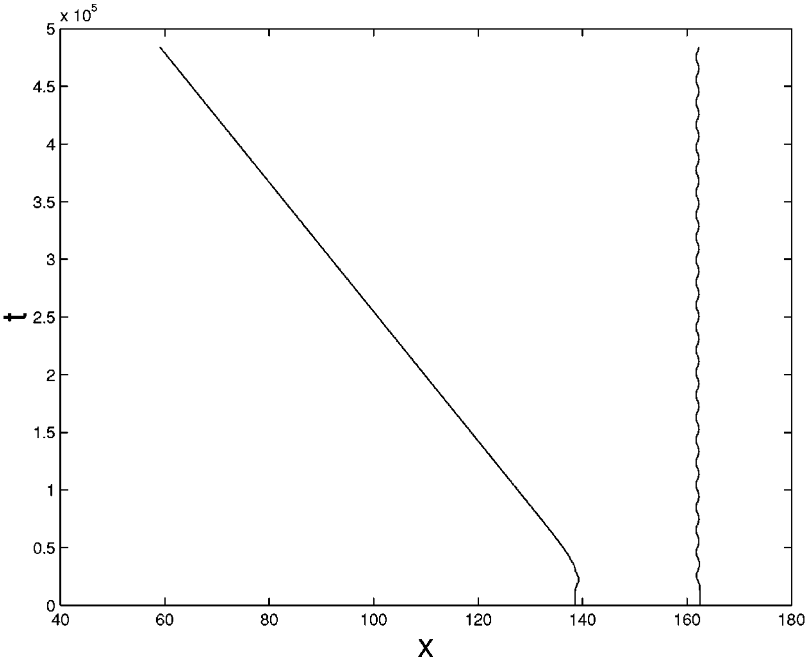

FIG. 6. Simulations of an up-down steady state (with the initial condition perturbed by a small random noise) with two fundamental pulses placed close to each other. As explained in the text, this configuration gives rise to eventual separation of the two solitons.

stability of such pulses for perturbations of the continuum NLS. For example, Cai *et al.* [43] and Barashenkov and Zemlyanaya [44] discuss the stability of multiple pulses to the driven and damped NLS. In particular, in both works the authors use the idea of deriving an effective potential of interaction between widely separated solitons to determine the stable configurations. Afanasjev *et al.* [45,35] considered the existence and stability of multiple pulses for Ginzburg-Landau type perturbations of the continuum NLS (also see [46] for existence results).

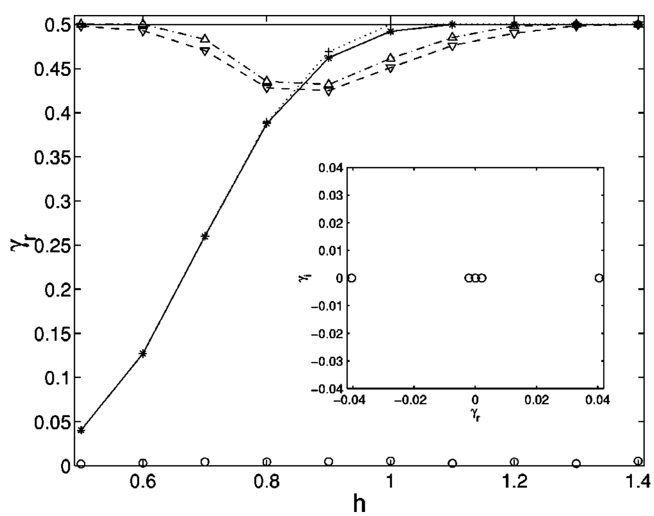

FIG. 7. Trajectories of the localized eigenmodes' eigenvalues as a function of the lattice spacing h for large values of h for which the local minimum up-down configuration is linearly stable. The circles indicate the tail-interaction eigenvalues, the stars and pluses (also dotted and solid lines, respectively) show the frequencies of the two Goldstone-mode pairs, while the up and down triangles (dash-dotted and dashed lines, respectively) indicate the frequencies of the modes bifurcating from the phonon band edge. The band edge is shown by the horizontal solid line at $\gamma_r = 0.5$.

In this work we have shown that multiple pulses in DNLS cannot be stable when pulses of the same phase are adjacent to each other. Since the criterion used in the proof only provides a necessary condition for stability, we then restricted our study to the two-pulse case and used variational methods to determine the local extrema (equilibrium points) of the effective energy as a function of the centers of the pulses. Such extrema were shown to exist for configurations where the multipulse center of mass is centered on a site or between sites. Saddle-point configurations have also been identified. Numerical methods were then used to complement the analy-

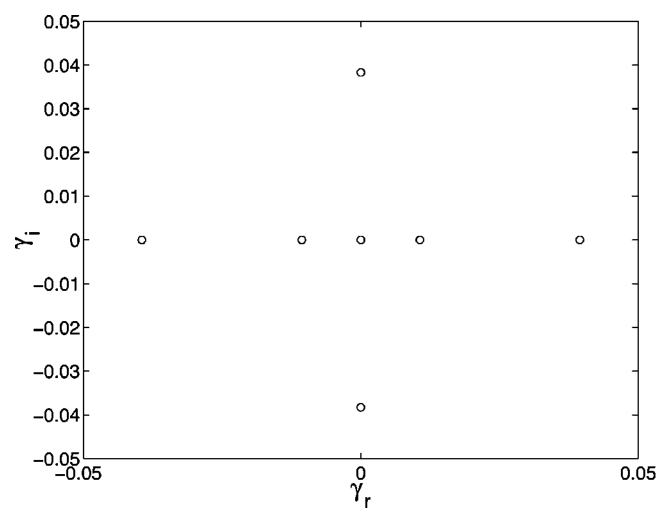

FIG. 8. A plot of the spectral plane (γ_r, γ_i) for the case of a saddle up-down configuration. The real tail-mode eigenvalue is closest to the origin (there are two zero frequencies existing due to the system's symmetries, as mentioned in the text). One pair of the Goldstone modes is stable (it has real eigenvalues), while, due to the saddle nature of the configuration, the second pair of Goldstone eigenvalues is imaginary, justifying the theoretically shown instability of such a configuration (see the text).

sis and in fact to demonstrate that linearly stable two-pulse configurations are possible, when there is a change of phase between the pulses and an appropriate (stable effective) equilibrium condition is satisfied.

It is important to construct a rigorous and more general theoretical framework for the calculations presented in Sec. III. It is an interesting open problem to determine when, and under what conditions, the effective Hamiltonian will yield the correct result. This problem has recently been considered by Kapitula [47] for the case of single pulses. It is anticipated that, when considering the interaction of widely separated primary pulses, the theory presented in [47] can be extended. Research into this question is currently in progress.

Furthermore, one needs a rigorous theory to explain the numerical observations presented in Sec. IV. For example, in Sec. II it was shown that the instability for the up-down solution could arise through the presence of either a purely real positive eigenvalue or a quadruplet of complex eigenvalues with a nonzero real part. However, the numerical simulations indicate that the second scenario does not occur for widely separated pulses. Work aiming to explain this feature is currently in progress.

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